Bulk viscosity and spectral functions in QCD

Guy D. Moore and Omid Saremi

Department of Physics, McGill University,
3600 University St., Montréal, QC H3A 2T8, Canada
E-mail: guymoore@physics.mcgill.ca, omid@physics.mcgill.ca

ABSTRACT: We examine the behavior of the spectral function for the T^μ_μ operator in QCD in the two regimes where it is possible to make analytical progress; weak coupling, and close to a second order QCD phase transition. We determine the behavior of the bulk viscosity in each regime. We discuss the problem of analytic continuation of the (lattice) Euclidean correlation function to determine the spectral function. In each case the spectral function has a narrow peak at small frequency; its shape would be challenging to extract accurately from lattice data with error bars.

KEYWORDS: Lattice QCD, Thermal Field Theory, Hadronic Colliders, QCD.

Contents

1. Introduction

Ongoing experiments at RHIC are exploring the QCD plasma, achieving initial temperatures which exceed the QCD phase transition (or crossover) temperature. Since the ions are large compared to the intrinsic QCD scale, they create a "macroscopic" sample of plasma, which allows the exploration of QCD at these temperatures in its hydrodynamic regime. The experimental program indicates that at central rapidities the plasma indeed behaves hydrodynamically, developing rather large radial and elliptic flows [1]. Viscous hydrodynamic analyses [2] only describe the observed elliptic flow well if the viscosity is very small, $\eta/s < 0.3$. Radial flow is caused by outward pressure accelerating the outer layers of the "fireball" radially. It builds up through the whole history of the fireball expansion until the hadrons become too dilute to interact. It is therefore sensitive to the whole time history of the plasma after the collision. It is sensitive to the equation of state, since this sets the relation between pressure and energy density. It should also be sensitive to bulk viscosity. The definition of bulk viscosity is a drop in the pressure, relative to the equilibrium value at the same energy density, due to expansion:

$$P = P_{\text{equil.}} - \zeta \, \boldsymbol{\nabla} \cdot \boldsymbol{v} \tag{1.1}$$

with $\boldsymbol{v}$ the flow velocity and ζ the bulk viscosity (and with all variables measured in the instantaneous local rest frame). Since the bulk viscosity reduces the outward pressure, it lowers the amount of radial flow. Of course, under a single set of experimental conditions the bulk viscosity may approximately mimic a modification in the equation of state; but comparing collisions of different centralities or of different sized nuclei, for which $\boldsymbol{\nabla} \cdot \boldsymbol{v}$ will vary, it should in principle be possible to isolate bulk viscosity from the equation of state.

At high temperatures where the coupling is weak and the theory is nearly conformal, the bulk viscosity is expected to be small [3–5]. Near the phase transition or crossover, however, it may be appreciable. This has encouraged a recent reanalysis of the bulk viscosity in QCD. In particular, Kharzeev and Tuchin have recently argued [6] that the bulk viscosity can be determined from lattice results for the equation of state (particularly

the temperature dependence of the pressure P and energy density ϵ) by using an exact sum rule and an *Ansatz* for the functional form of the spectral function of stress-stress correlations. And Meyer [7] has performed a (pure-glue) lattice study of the Euclidean T^μ_μ correlation function, with a view towards an analytic continuation of the lattice data to determine the real-time spectral function and in particular the bulk viscosity.

Using Euclidean data to reconstruct real-time correlation functions is in general ill-posed without some assumptions about the shape of the real-time correlation function. Therefore we feel that it is useful to learn whatever we can about the spectral function for T^μ_μ in whatever regimes analytic information is available. We have found two such regimes. At weak coupling (high temperature) we can compute the spectral function perturbatively. And near a second order phase transition (such as probably occurs in realistic QCD at some point in the T–μ plane), we can make reliable statements about scaling behaviors based on universality arguments. The latter case may be of importance for real-world QCD; if the critical point is close enough to the temperature axis, heavy ion collisions may explore a near-critical crossover, with long correlation lengths and sensitive T dependence of thermodynamical variables.

As discussed above, the bulk viscosity is defined as a deviation of the pressure $P = \frac{1}{3}T^i_i$ from its equilibrium value due to expansion. The operator which generates expansion of the system is also $\frac{1}{3}T^i_i$, and as Kubo showed, one can treat a slowly expanding system by coupling this operator to an external source. The bulk viscosity is determined as the linear response of the operator $\frac{1}{3}T^i_i$ to such an external source [8]:

$$\zeta = \frac{1}{2}\lim_{\omega\to 0^+}\frac{1}{\omega}\int_{-\infty}^{\infty}dt e^{-i\omega t}\int d^3x \left\langle \left[\tfrac{1}{3}T^i_i(x,t)\,,\,\tfrac{1}{3}T^i_i(0,0) \right] \right\rangle. \tag{1.2}$$

Because the energy $\int d^3x T^{00}$ is conserved, it is harmless to shift T^i_i by the energy or any multiple of the energy in the above. Two useful choices are k such that $\langle T^i_i + kT^0_0\rangle = 0$ (so the operator we use has vanishing expectation value) and $k = 1$, so the correlator is replaced with a correlator of T^μ_μ.[1]

The purpose of this paper is to analyze the spectral function

$$\rho(\omega) \equiv \int dt e^{-i\omega t}\int d^3x \frac{1}{9}\left\langle \left[T^\mu_\mu(x,t)\,,\,T^\nu_\nu(0,0) \right] \right\rangle \tag{1.3}$$

both in the weak coupling regime and close to the phase transition point, focusing on low frequencies. The bulk viscosity is determined by $\zeta = \frac{1}{2}\lim_{\omega\to 0}\rho(\omega)/\omega$. Section 2 examines the perturbative regime, generalizing the bulk viscosity calculation of [5] to nonzero frequencies. We show that ρ/ω has a peak at zero frequency of height $\mathcal{O}(\alpha_s^2 T^3)$ and area $\int d\omega \rho/\omega \sim \alpha_s^{7/2}T^4$. Both the width and area under the peak contradict the Kharzeev-Tuchin study [6]. Section 3 studies the spectral function and bulk viscosity near a second order phase transition. Drawing on work by Onuki [9], we argue that the bulk viscosity shows a *power divergence* as the critical point is approached; along the crossover line a distance t in the T,μ plane from the critical point, $\zeta \propto t^{-z\nu+\alpha}$, with $\nu = .630$ the scaling

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[1]We use $[-+++]$ metric conventions.

exponent for the correlation length, $\alpha = .110$ the critical exponent for the heat capacity, and $z \simeq 3$ the dynamical critical exponent. Again, this corresponds to a sharp peak in ρ/ω near zero frequency. This result contradicts ref. [10]. We end with a discussion section in which we examine the implications of our results for the program of determining the spectral function via analytic continuation of Euclidean data.

2. Spectral function at weak coupling

For simplicity we will consider only pure glue QCD here; the behavior of QCD with quarks of negligible mass is more complicated but qualitatively the same. The trace of the stress tensor is the generator of dilatations, which are a classical symmetry of the theory. However this symmetry is broken at the quantum level: under dilatations the action $S = \int d^4x \frac{1}{2g^2} \text{Tr}\, G_{\mu\nu}G^{\mu\nu}$ changes because the inverse gauge coupling $1/2g^2$ changes by $-\beta/g^4$, with $\beta = \frac{\mu^2 d}{d\mu^2}g^2 \sim g^4$ the beta function for the coupling g^2. Therefore T^μ_μ can be replaced with $T^\mu_\mu \to \frac{\beta}{g^4} \text{Tr}\, G^2$ the square of the field strength [11]. The correlation function eq. (1.3) is therefore $4\beta^2/g^4$ times the correlation of the Lagrangian density with itself.

The spectral function is related to the Wightman function by a KMS relation:

$$G^>(\omega) \equiv \int dt e^{-i\omega t} \int d^3x \frac{1}{9} \langle T^\mu_\mu(0,0) T^\nu_\nu(x,t) \rangle_{\text{conn}} \,,$$

$$G^>(\omega) = \frac{1}{1 - e^{-\omega/T}} \rho(\omega) \,. \tag{2.1}$$

It is convenient to shift the operator T^μ_μ by a multiple of T^0_0 such that its expectation value vanishes, so that the connected correlation function is the same as the full correlation function. In our case this means that we need to work not in terms of T^μ_μ but in terms of $\mathcal{O} \equiv T^\mu_\mu - T^0_0 \langle T^\nu_\nu \rangle / \langle T^0_0 \rangle \simeq T^i_i + 3v^2_s T^0_0$, with v_s the speed of sound. This shift to the operator is $\mathcal{O}(g^4)$ and will only be important for some terms in what follows. In fact the only difference between using $\mathcal{O}$ and using T^μ_μ is that the T^μ_μ correlation function will have an extra delta function strictly at zero frequency and of height equal to the energy susceptibility (heat capacity) times $(\langle T^\mu_\mu \rangle / \langle T^0_0 \rangle)^2$. The Wightman correlator for $\mathcal{O}$ is missing this delta function; its value at $\omega = 0$ is $2T$ times the bulk viscosity.

Perturbatively the leading contribution to this correlation function is that of figure 1. This gives rise to a leading order contribution (defining[2] $Q = (\omega, 0)$) of

$$G^>(Q) = \frac{2\beta^2(g)d_A}{9g^4} \int \frac{d^4P\, d^4R}{(2\pi)^4} \delta^4(Q - P - R)\, G^>_{\mu\alpha}(P) G^>_{\nu\beta}(R)$$
$$\times (g^{\mu\nu} P \cdot R - P^\mu R^\nu)(g^{\alpha\beta} P \cdot R - P^\alpha R^\beta)\,, \tag{2.2}$$

where $d_A = N^2_c - 1$ is the dimension of the group. Note that both gauge boson propagators are Wightman (cut) propagators, given by

$$G^>_{\mu\nu}(P) = [n_b(p^0) + 1]\, 2\pi\delta(P^2 + m^2) \sum_\lambda \epsilon_\mu(\lambda)\epsilon^*_\nu(\lambda)\,, \tag{2.3}$$

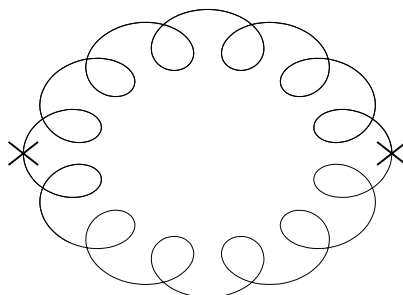[2]Capital letters are 4-vectors; lower case are their spatial components.

Figure 1: Graph which dominates G^2–G^2 correlations at weak coupling. The $\times$ symbols represent the G^2 operator insertions.

with λ the polarization state; $\epsilon_\mu P^\mu = 0$. The use of massive dispersion relations[3] accounts for forward scattering in the plasma; the plasma mass is $m^2 = m_{\rm D}^2/2 = g^2 N_{\rm c} T^2/6$. Since we are working perturbatively, we will treat $m_{\rm D}^2 \ll T^2$ the energy scale of typical quasiparticle excitations and we will ignore longitudinal gluons, which have an exponentially small spectral weight for $p^0 \sim T$ [12].

There are two contributions to $G^>(Q)$, corresponding to the two ways the conditions $\delta^4(Q-P-R)\delta(P^2+m^2)\delta(R^2+m^2)$ can be satisfied at vanishing $\boldsymbol{q}$. Since $|\boldsymbol{p}| = |\boldsymbol{r}|$, we have $p^0 = \pm q^0$. The contribution where $p^0 = r^0 = \omega/2$ gives rise to a continuous contribution, corresponding to a cut in the retarded function:

$$G^>_{\rm cut} = \left[n_b\!\left(\frac{\omega}{2}\right) + 1 \right]^2 \frac{2\beta^2(g)}{9g^4} \frac{2d_{\rm A}\,\omega^4}{32\pi} \quad \propto g^4\omega^4\,. \tag{2.4}$$

Here $2d_{\rm A}$ is the number of color and spin states of gluons. For this contribution, the difference between $\mathcal{O}$ and T_μ^μ is an $\mathcal{O}(g^2)$ correction and can be neglected. The parametric behavior is simple to understand: g^4 arises because the trace anomaly makes T_μ^μ naturally $\mathcal{O}(g^2)$ and we are computing the correlator of two T_μ^μ's; and the ω^4 behavior follows on dimensional grounds. Note that for $\omega < T$ the behavior changes to $g^4\omega^2 T^2$ because of the statistical functions, and is further modified below $\omega \sim gT$.

The other contribution arises when $p^0 = -r^0$. This requires $\omega = 0$, and therefore corresponds to a delta function in the Wightman function (pole in the retarded function). If we evaluate the correlation function for T_μ^μ operators without subtracting disconnected contributions, we find

$$G^>_{\rm pole}[\text{inc. disconnected}] = \delta(\omega)\frac{2}{9g^4} \int \frac{p^2 dp}{4\pi E_p^2}\, 2d_{\rm A}\,(\beta\, P^2)^2\, n_b(p^0)[1+n_b(p^0)]\,. \tag{2.5}$$

This would vanish were it not for dispersion corrections for hard gluons, mentioned above: $P^2 = -m^2 = -m_{\rm D}^2/2 \sim g^2 T^2$. Inserting this estimate and taking $p \sim T$, one finds that the delta-function contribution naively scales as $g^8 T^5 \delta(\omega)$, with the g^8 arising as two powers of g^2 from the beta functions and two powers of g^2 because of the dispersion relations.

Since this extra g^4 suppression may come as a surprise for some readers, we will pause to discuss its physical origin. Physically, at zero frequency and momentum the

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[3]We use $[-+++]$ metric convention

$T^\mu_\mu \propto G^{\mu\nu} G_{\mu\nu}$ operator is probing thermal excitations without disturbing them. A massless gauge excitation has equal E^2 and B^2, and therefore $G^{\mu\nu} G_{\mu\nu} = B^2 - E^2$ vanishes for an undisturbed, propagating gauge boson. It is only because of plasma dispersion corrections, which allow $E^2 \neq B^2$, that the cancellation is not exact. Therefore the pole contribution "wants" to vanish for two reasons; the smallness of the beta function, contributing $\beta^2/g^4 \sim g^4$, and the smallness of dispersion corrections, yielding $(B^2 - E^2)^2 \sim (g^2)^2 \sim g^4$.

However there are some complications in evaluating this contribution. First, $\beta P^2 = -\beta m^2$ is the same order as $\langle T^\mu_\mu \rangle$. Therefore the result above is contaminated by disconnected parts, and we should evaluate the correlation function using the operator $\mathcal{O}$ defined above. This amounts to the substitution $-\beta m^2 \to (v_s^2 - \frac{1}{3})p^2 - \beta m^2$. The contribution to the Wightman correlator is approximately

$$G^>_{\text{pole}} = \delta(\omega) \frac{2}{9} \, 2d_A \frac{1}{4\pi} \int_0^\infty n(p)(1+n(p)) \left[\left(\frac{1}{3} - v_s^2 \right) p^2 + \frac{\beta m_D^2}{2g^2} \right]^2 \frac{p^2 \, dp}{E_p^2} \,. \tag{2.6}$$

The next complication is that this integral is infrared singular, indicating that the "pole" contribution to the Wightman function is dominated by soft physics. This happens because B^2 and E^2 come further from canceling as one considers more infrared, and therefore more dispersion-corrected, excitations. The infrared singular, $\sim \int dp/p^2$, behavior is cut off at the scale $p \sim gT$, where E_p^2 deviates strongly from p^2. A complete treatment of this regime requires a detailed analysis using Hard Thermal Loop (HTL) effective theory [12] and we will not attempt it here. However we can easily see that the linear divergence, cut off at the gT scale, reduces by 1 the power of g appearing in the height of the "delta function" contribution, such that $\int_{-gT}^{gT} d\omega G^>(\omega) \sim g^7 T^5$. Physically, this is because, in the soft $p \sim gT$ region, there is no relation between E^2 and B^2, so the extra g^4 suppression found above is absent for such excitations; however they represent only a g^3 fraction of the energy density, leading to a delta function contribution $\sim g^3 \times g^4 \sim g^7$ (the g^4 still arising from the square of the beta function).

The last complication is that the "delta function" we just found is not really of zero width; interactions broaden it into a sharp peak. Since the bulk viscosity is determined by the height of this peak, we need to determine its shape. Interactions mean that the delta-function behavior found above receives corrections. The delta function arises from integrating over all (thermal) momenta which can run in the loop in figure 1. Very roughly, interactions mean that each particle running in the loop contributes not a delta function but a Lorentzian of the same area, with width Γ set by the large angle or large momentum change scattering rate for this particle. The width Γ is momentum dependent and is parametrically $\Gamma \sim g^4 T^3/p^2$ (see [13] for a discussion of the relevant scattering processes). Therefore, although soft $p \sim gT$ particles dominate the area of the peak, it is hard $p \sim T$ particles which dominate its height, since they have narrower widths. At frequency $g^2 T \gg \omega \gg g^4 T$ the dominant p is $p \sim T(g^4 T/\omega)^{1/2}$. The correlation function is parametrically

$$G^>(\omega) \sim \begin{cases} g^4 T^4 \,, & \omega \lesssim g^4 T \\ g^6 T^4 \sqrt{T/\omega} \,, & g^2 T \gg \omega \gg g^4 T \end{cases} \,. \tag{2.7}$$
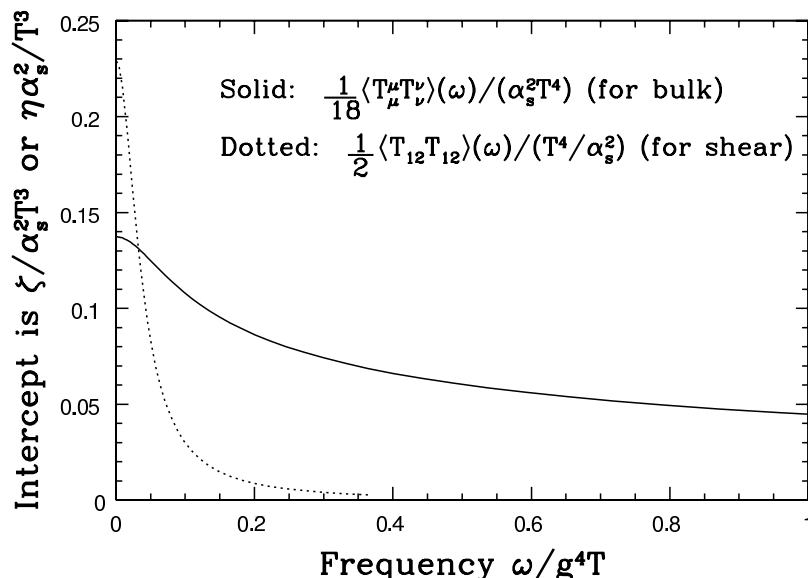
Figure 2: Wightman correlators at small frequency for the operators T^μ_μ (relevant for bulk viscosity) and $T_{ij} - \frac{\delta_{ij}}{3} T_{kk}$ (relevant for shear viscosity) for $N_c = 3$ pure glue QCD at weak coupling (and setting $m_D = 1.5T$). The "peak" in the bulk correlation function is wider and has much larger "shoulders," as discussed in the text.

For $\omega \gtrsim g^2 T$ contributions from the Landau cut and higher order diagrams cannot be neglected. We will not attempt to address this region here.

We can make this estimate more precise by extending the results on bulk viscosity [5] (strictly zero ω) to treat $\omega \sim g^4 T$ using the approach of ref. [14]. The idea is that the correlator $G^>(\omega)$ for $\omega \ll g^2 T$ is essentially determined by kinetic theory, that is, by solving a Boltzmann equation. For a detailed diagrammatic justification for this fact (in the context of scalar field theory) see [4]. The Boltzmann equation can be solved variationally using the tools developed in ref. [15] for zero frequency and applied to bulk viscosity in [5]. Ref. [14] showed how to extend these tools from zero to small frequency in the context of current-current correlation functions, and there is no problem doing so for stress tensor correlation functions too. The computed shape of the correlator $G^>(\omega) \simeq T\rho(\omega)/\omega$ for $\omega \sim g^4 T$ is displayed in figure 2, which also compares the shape of the "peak" in the T^μ_μ correlation function to the peak in the shape of the $T_{ij} - \frac{\delta_{ij}}{3} T_{kk}$ correlation function relevant for shear viscosity. Besides the parametrically large difference in the heights of the peaks (already discussed above), the figure shows that the peak in the T_{ij} correlator is much narrower, more closely resembling a Lorentzian. This is because, unlike the T^μ_μ correlator, the T_{ij} correlator is not sensitive to soft excitations.

Let us compare this weak-coupling behavior to that claimed in a recent analysis by Kharzeev and Tuchin [6]. They derive what they claim to be an exact sum rule showing that the frequency integral of the spectral function is related to the temperature dependence of

the energy density and pressure:

$$\text{Eq. (12) of [6]:} \quad \int_{-\infty}^{\infty} \frac{\rho(\omega)}{\omega} d\omega = T^5 \frac{\partial}{\partial T} \frac{\epsilon - 3P}{T^4} + 16 E_{\text{vac}}\,. \tag{2.8}$$

The vacuum energy contribution can be removed by replacing the spectral function with its thermal part. They also state that the perturbative contributions should be subtracted, though it is not clear to us whether this refers to the righthand, lefthand, or both sides of the relation.

First we analyze the righthand side. At weak coupling and in the absence of quark masses it has long been known that P scales as T^4 with a coefficient which can be expanded in powers of g:

$$P(T) = T^4 \times \left(A + Bg^2[\mu] \left(1 + \frac{\beta}{g^2} \ln \frac{T^2}{\mu^2} \right) + \mathcal{O}(g^3) \right). \tag{2.9}$$

The coefficients A, B are known [16] but their precise values are not important to this discussion. (In fact the coefficients through order $g^6 \ln(g)$ are also known [17]). All that matters to us here is the functional form and that the scale dependence of g^2 is set by the temperature T, which we have shown explicitly in the above by including the $\mathcal{O}(g^4)$ term with an explicit log which renders the result μ independent. Using that $\epsilon = T dP/dT - P = 3P + 2T^4 B\beta$, one easily finds that

$$v_{\text{s}}^2 = \frac{dP}{d\epsilon} = \frac{1}{3} - \frac{2B}{9A}\beta + \mathcal{O}(g^5) \sim g^4\,, \qquad \text{and} \qquad \frac{\epsilon - 3P}{T^4} = 2B\beta \sim g^4\,, \tag{2.10}$$

where the coupling g appearing in the expression for β should be understood as being evaluated at $\mu \sim T$. The dominant T dependence is from this renormalization dependence of g:

$$\frac{Td}{dT} \frac{\epsilon - 3P}{T^4} = 2B \frac{Td}{dT}\beta = \frac{8B}{g^2} \beta^2 \sim g^6\,. \tag{2.11}$$

Therefore a literal interpretation of the righthand side of eq. (2.8) leads to a parametrically $\mathcal{O}(g^6 T^4)$ result.

On the other hand, we can insert our analytical results for the spectral function into the lefthand side of eq. (2.8). The nearly-delta function peak gives a contribution of order $g^7 T^4$ (see the discussion after eq. (2.6), and use that $\rho/\omega \simeq G^>/T$). The cut contribution, after subtracting the vacuum contribution,[4] is $\mathcal{O}(g^4 T^4)$. Therefore a literal interpretation of both sides of the equation leads to an inequality. And subtracting the rising cut contribution does not help, since it changes the l.h.s. from $\mathcal{O}(g^4)$ to $\mathcal{O}(g^7)$, not g^6. Therefore Kharzeev and Tuchin's result can only make sense if some subtraction is implied on both sides of the equation, in which case it has no utility in the perturbative regime considered here.

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[4]After vacuum subtraction, the leading-order cut contribution decays exponentially. However, subleading in g^2 contributions do not. According to S. Caron-Huot [18], the leading thermal corrections at large ω scale as $\rho_T \sim g^2 \rho_{\text{vac}} (T/\omega)^4$ both for current-current and stress-stress correlation functions, see also [19, 20].
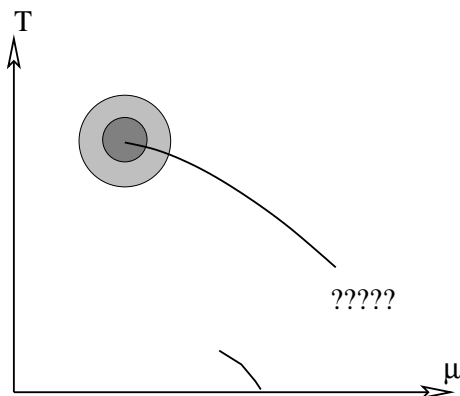
Figure 3: Expected phase structure in the $T - \mu$ plane for realistic QCD. Along the T axis there are no phase transitions, but at finite chemical potential there is a 1'st order transition line which terminates at an Ising universality class critical point. Criticality can give parametric predictions for behavior near this point, roughly in the shaded region. At large chemical potential and small temperature there is a 1'st order transition associated with nuclei and possibly other transitions associated with color superconductivity, which are not important here.

3. Spectral function near the 2'nd order transition point

There is another regime in which it is possible to say something analytical about the T^μ_μ spectral function. Close to a second order phase transition, low frequency and momentum correlation functions are dominated by long wavelength fluctuations which obey universal thermodynamical properties which allow for a scaling analysis. In other words, we can use universality arguments to determine the functional form of correlation functions, such as their parametric dependence on the difference between the temperature and the equilibrium temperature. Furthermore, hydrodynamic arguments make it possible to extend the universality predictions to dynamical (unequal time) correlation functions, though this requires some additional information about what quantities are conserved [21].

It is believed that the phase diagram (in the T-μ plane) for realistic QCD with two light (but not massless) and one fairly light quark flavor possesses a first order phase transition line which terminates in an Ising universality class second order endpoint [22], as illustrated in figure 3 (see however [23]). The plasma generated in very high energy heavy ion collisions is expected to follow a trajectory close to the T axis (small μ), which probably misses the phase transition line [24] but may nevertheless experience a rather sharp crossover with a large correlation length. In principle, intermediate energy heavy ion collisions may explore the critical point [25]. In any case, even if it cannot be directly probed experimentally, it is interesting to consider the vicinity of the critical point to see what general lessons it teaches us about T^μ_μ correlation functions. Also note that the pure-glue theory is expected to have a first-order phase transition point [26] but the transition is very weak with a long correlation length [27] and so scaling arguments might be suggestive here as well (and two color QCD should have a second order transition in the Ising universality class [26]).

Near the critical point, some linear combinations of the temperature and chemical

potential map to the temperature and magnetic field variables of the Ising model. Ordinary fluids also have a phase transition between liquid and gas phases with a critical point in the Ising universality class. There is also a mapping of the Ising variables to the temperature and pressure variables of this system, and therefore a mapping between T, μ in QCD and T, P in the liquid-gas system, as illustrated in figure 4.

We need more information to extend universality arguments to unequal time correlation functions [21]. The long range fluctuations in the order parameter are coupled to microscopic degrees of freedom which should lead to diffusive (Langevin) dynamics for the order parameter fluctuations. The order parameter should therefore admit a Ginsburg-Landau type description, both thermodynamically and dynamically. This leads as usual to universality in the behavior of static correlation functions between all critical systems with the same dimensionality and underlying symmetries. But at the level of unequal time dynamics, the Ginsburg-Landau description must also include any locally conserved quantities. In particular, in QCD the temperature and chemical potential are dual to energy density and baryon number density, which are both conserved. Since T and μ map to linear combinations of the Ising variables t and h, QCD behaves like an Ising system with a locally conserved energy and magnetization. A local, upward fluctuation in the magnetization would bias the fluctuations in the order parameter to be positive in that neighborhood. Similarly, an upward fluctuation in the (Ising) energy would bias fluctuations of either sign to be smaller. These local net modifications would persist as long as the density of the responsible conserved quantity remained in that neighborhood. Therefore there are correlations in the order parameter which persist as long as the conserved quantities retain their local values. But conserved quantities cannot relax locally; they have to move away, which on large length scales occurs diffusively. And the diffusion of these conserved quantities is in turn sensitive to the order parameter fluctuations, leading to a coupled problem. Fortunately, there is still a notion of universality; systems with second order phase transitions in the same (static) universality class, and which have the same set of conserved quantities, will show the same dynamical scaling behavior near the transition point [21].

For realistic QCD the conserved quantities are the 4-momentum and baryon number. The dynamical universal behavior is therefore the same as for the liquid-gas phase transition [28], since this system also has a conserved energy, momentum, and particle number.

Fortunately Onuki has performed a detailed study of the dynamical critical behavior of the liquid-gas system near its critical point, focusing on the behavior of bulk viscosity [9]. Since the dynamical universalities are the same, this can be directly adapted to the QCD case.[5] As one approaches the transition point, the correlation length ξ exceeds the natural microscopic length scale l_{mic} ($l_{mic} = T^{-1}$ in a relativistic setting). The dynamics of the order parameter fluctuations on a scale $l_{mic}^{-1} < k < \xi^{-1}$ are dissipative and slowly evolving,

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[5]It may seem confusing that a relativistic theory, QCD, and a nonrelativistic theory, conventional fluids, should display the same dynamical criticality. But remember that at the hydrodynamical level, all but one degree of freedom in either system evolves dissipatively, so propagation of hydrodynamical quantities is slow (nonrelativistic) in either case. The exception is sound waves, which propagate at a fixed velocity $c_s < 1$ in each theory and which turn out not to be important to critical behavior [21].
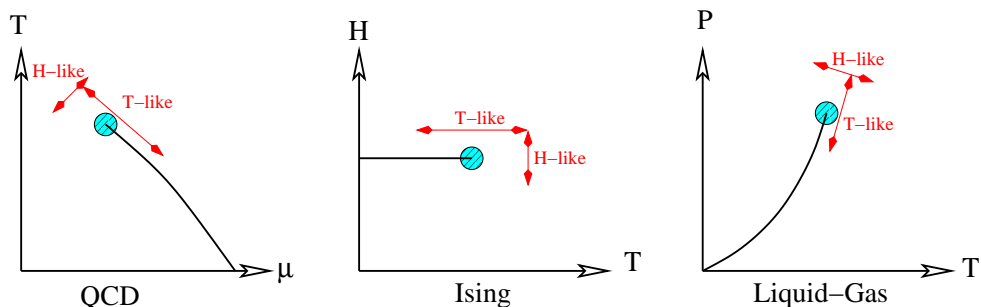
Figure 4: (Color online) Relation between the Ising model temperature and magnetic field directions and the directions in QCD and the liquid-gas system.

with the fluctuations changing on a time scale $\tau_k \propto k^{-z}$, with z the dynamical critical exponent:

$$\int d^3x e^{i\mathbf{k}\cdot\mathbf{x}}\langle\psi(x,\tau)\psi(0,0)\rangle = \chi(k)e^{-\tau/\tau_k}\,, \qquad \tau_k \sim k^{-z}\,, \tag{3.1}$$

with ψ the order parameter and $\chi(k)$ its momentum-dependent susceptibility. In 3D Ising systems with liquid-gas dynamical universality, $z \simeq 3$ [21]. Equilibration is dominated by these slow modes.

In the Ising model, the free energy F is a function of the reduced temperature $t = (T - T_c)/T_c$ and the magnetic field h. Close to the transition point it behaves as

$$F(t,h) = F_{\text{nonsing}}(t,h) + t^{2-\alpha}F_{\text{sing}}(t/|h^{1/(\beta+\gamma)}|)\,. \tag{3.2}$$

Here F_{sing} is a contribution to the free energy arising from the long range (near)critical fluctuations. Note that at small t these fluctuations give almost no contribution to the pressure $P = -F$. The entropy is $S = -\partial F/\partial t$ and $E = ST + F$. Therefore the heat capacity $C_v = \partial E/\partial T$ behaves as $C_v = C_{v,\text{nonsing}} + C_{v,\text{sing}}t^{-\alpha}$, displaying a weak divergence as $t \to 0$ which arises from the long range fluctuations in the order parameter. These fluctuations provide no pressure but dominate the heat capacity, leading to a speed of sound $\partial P/\partial\epsilon \simeq 0$. The same is true in QCD near the critical point except that the Ising "temperature" direction corresponds to a linear combination of temperature and chemical potential (so the "heat capacity" referred to here is really a linear combination of heat and baryon number capacity).

A small, rapidly applied compression will not promptly change the long-range correlations of the order parameter; instead it changes the noncritical degrees of freedom, and therefore leads to an instantaneous pressure rise $\Delta P \sim \Delta\epsilon$ typical of relativistic degrees of freedom. Then, on a time scale $\tau \sim \xi^z$, the long range fluctuations equilibrate, absorbing almost all of $\Delta\epsilon$ (since they dominate the heat capacity) and allowing the pressure to relax to the equilibrium value. Therefore one might guess that $\zeta \sim \xi^z$. [In the next few paragraphs, dimensions in parametric estimates are to be filled in with the appropriate power of the intrinsic scale T.] Along the "crossover line", meaning the map of the $h = 0$, $t > 0$ line in the Ising system, this behavior is $\zeta \sim t^{-z\nu}$, with t the distance in the T,μ plane from the critical point and ν the critical exponent $\nu \simeq 0.630$ in the Ising system.

In fact a more detailed analysis [9] shows that $\zeta \sim \xi^{z-\alpha/\nu} \sim t^{-z\nu+\alpha}$. Onuki has given a careful derivation of this result; here we give a simple intuitive explanation of why it is true. Consider a small, rapid compression. After an intermediate amount of time τ satisfying $T^{-1} < \tau < \xi^z$, all modes with $k > \tau^{1/z}$ have equilibrated; those with smaller k (longer wavelength) remain out of equilibrium. The heat capacity represented by the equilibrated long-range fluctuations is $C_{v,k<\tau^{1/z}} \sim k^{\alpha/\nu} \sim \tau^{\alpha/z\nu}$. Therefore a fraction $\tau^{-\alpha/z\nu}$ of the energy remains in the noncritical degrees of freedom so the pressure is elevated with respect to equilibrium by $\Delta P \sim \Delta\epsilon\tau^{-\alpha/z\nu}$. This is true at all times $\tau < \xi^z \sim t^{-z\nu}$. The bulk viscosity is found by integrating this result over all time; it is dominated by $\tau \sim \xi^z$, and gives $\zeta \sim \xi^{z-\alpha/\nu}$. Fourier transforming the τ dependence gives the spectral function (at low frequencies); the Wightman function for frequency ω is dominated by times $\tau \sim \omega^{-1}$ and is parametrically $G^>(\omega) \sim \omega^{-1+\alpha/z\nu}$. Reinserting powers of T, we find

$$G^>(\omega) \sim T^3(\omega/T)^{-1+\alpha/z\nu} \quad [\omega > \xi^{-z}T^{1-z}]; \qquad \zeta \sim T^3(\xi T)^{z-\alpha/\nu}. \qquad (3.3)$$

Once again, we find that the spectral function [divided by frequency] possesses a narrow peak for $\omega < T$. Note that the total area under the peak is finite and is essentially determined by noncritical physics; however the height of the peak diverges as a power of the distance to the critical point: $\zeta \sim t^{-z/\nu+\alpha}$ as one approaches the critical point along the crossover line.

4. Discussion

In both of the cases where we can gain analytical insight (weak coupling and close to the second order phase transition point), the spectral function $\rho(\omega)/\omega$ has a peak at low frequency. This peak arises because there are degrees of freedom which equilibrate very slowly; the width of the peak corresponds to the inverse of the relaxation time towards equilibrium. At weak coupling, all degrees of freedom exhibit slow relaxation. In this case relaxation is slowest for high momentum quasiparticles, which dominate the height of the peak, though low momentum fluctuations dominate the area under the peak. Near the phase transition point, it is long range fluctuations in the order parameter which equilibrate slowly and give rise to the peak in the spectral function. These fluctuations are important to bulk viscosity because they dominate the heat capacity, though they are of little importance for shear viscosity since they contribute almost nothing to the pressure.[6] The bulk viscosity is determined by the dynamical critical exponent which determines the critical slowing down of these fluctuations.

It would be too bold to extrapolate from these two examples to claim that $G^>$ or ρ/ω always has such a low frequency peak. For instance, at $2T_c$, where there are no near-critical fluctuations but the coupling is strong, it is quite possible that all degrees of freedom equilibrate quickly and the spectral function is smooth near zero frequency. However, the most interesting temperature range experimentally is temperatures close to T_c, both

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[6]Note however that it is believed that shear viscosity also shows a weak divergence at the second order transition point for systems in the liquid-gas dynamical universality class; roughly $\eta \sim t^{-0.05}$ [21].
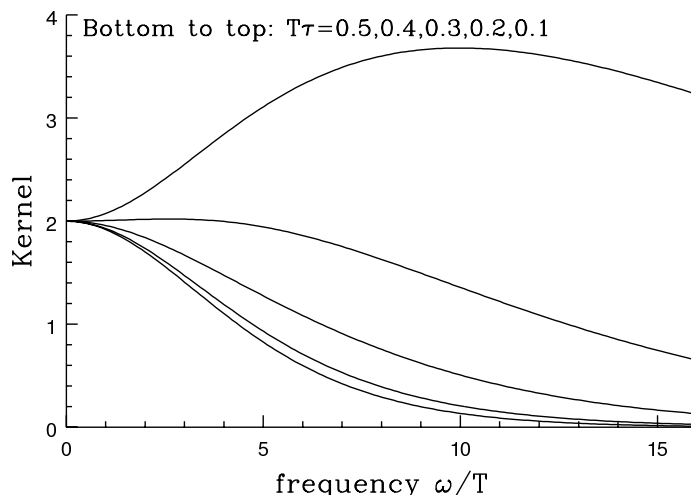
Figure 5: Kernel K relating the (Minkowski) spectral function to the (Euclidean) unequal τ correlation function, as a function of (Minkowski) frequency for some representative values of Euclidean time τ. The behavior at small ω is almost τ independent.

because the plasma almost surely explores such temperatures in real heavy ion collisions and because it is the only place where we expect the bulk viscosity to be appreciable.

The presence of a peak in $\rho(\omega)/\omega$ at small ω is problematic for the reconstruction of the bulk viscosity from unequal Euclidean time correlations measured on the lattice. The Euclidean time correlation function[7]

$$G_{\rm E}(\tau) \equiv \int d^3x \frac{1}{9} \langle T_{\rm E}^{\mu\mu}(x,\tau) T_{\rm E}^{\nu\nu}(0,0) \rangle_{\rm conn} \tag{4.1}$$

is related to the spectral function via the integral relation

$$G_{\rm E}(\tau) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\rho(\omega)}{\omega} K(\omega,\tau), \qquad K(\omega,\tau) \equiv \frac{\omega \cosh[\omega(\tau-1/2T)]}{\sinh(\omega/2T)}. \tag{4.2}$$

In principle, complete knowledge of $G_{\rm E}(\tau)$ as an analytic function allows for the reconstruction of $\rho(\omega)/\omega$. However in the lattice context one only has numerical data with error bars at a finite number of times τ, and some procedure (such as the Maximal Entropy Method [29]) must be used to reconstruct the spectral function.

Figure 5 shows the function $K(\omega,\tau)$ as a function of ω for some representative values of τ. This figure illustrates the challenge of learning about a sharp spike in the spectral function: all of the curves take the same value near zero frequency. In other words, each τ gives degenerate information about a peak at small frequency — the area under the peak — with almost no sensitivity to the shape of the peak. Therefore, details about the shape of such a peak effectively have to be inserted as part of the fitting procedure. Meyer's

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[7]It is important to compute the connected correlation function on the lattice-or equivalently to subtract off the mean value of each operator, which is equivalent at nonzero τ to rescaling the T^{00} contribution until the expectation value vanishes. We thank Derek Teaney for emphasizing this point to us.

recent results [7] effectively assume that $\rho(\omega)/\omega$ is smooth at small ω, which we have just shown is not a good assumption, at least near a second order transition or a very weak first order transition.

So what is the best way to estimate ζ in QCD near the transition (or crossover) point? There are two things which we *can* (in principle) determine reliably from a lattice calculation. One is the correlation length ξ for the order parameter ($\langle \bar{\psi}\psi \rangle$ for nearly chiral 2-flavor QCD). The other is the total area under the small momentum peak in the spectral function. This can be determined by measuring $G_E(\tau)$ on the lattice and fitting for the spectral function allowing a narrow peak at the origin, with the area under the peak used as a parameter in the fit. Call this area $G_\int = \int_{|\omega|<T} \rho(\omega)d\omega/2\pi\omega$. [We expect it to be almost as large as $G_E(\tau/2)$.] Then we know parametrically that

$$\zeta = A\frac{G_\int}{2T}(\xi T)^{z-\alpha/\nu} \, . \tag{4.3}$$

What we do not know is the coefficient A. This coefficient requires dynamical information; it represents the constant in the scaling relation between a wave number k and the relaxation time of fluctuations at that wave number, $T\tau_k = A(T/k)^z$. To get a phenomenological estimate of the bulk viscosity where (ξT) is large, we have to make some reasonable guess for the coefficient A. We advocate $A = 1$ and $A = \pi^z$ as two reasonable choices, based on the assumptions that the critical regime begins at the scale T and the scale πT respectively. Clearly there is a rather large band of uncertainty in the final determined ζ.

In summary, we have shown that in both of the cases where analytic methods can be brought to bear (weak coupling and close to the second order transition point), the spectral function needed to determine the bulk viscosity has a narrow peak at low frequency. At weak coupling the peak has a height $\lim_{\omega\to0}\rho(\omega)/\omega \sim g^4 T^4$ and area $\int d\omega\rho(\omega)/\omega \sim g^7 T^5$; near the critical point the area is $\sim T^5$ and the height diverges as $T^4(\xi T)^{z-\alpha/\nu}$. This behavior complicates the reconstruction of the spectral function from Euclidean data.

Acknowledgments

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